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Blackbody radiation, conformal symmetry, and the mismatch between classical mechanics and electromagnetism

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Abstract

The blackbody radiation problem within classical physics is reviewed. It is again suggested that conformal symmetry is the crucial unrecognized aspect, and that only scattering by classical electromagnetic systems will provide equilibrium at the Planck spectrum. It is pointed out that the several calculations of radiation scattering using nonlinear mechanical systems do not preserve the Boltzmann distribution under adiabatic change of a parameter, and this fact seems at variance with our expectations in connection with derivations of Wien's displacement theorem. By contrast, the striking properties of charged particle motion in a Coulomb potential or in a uniform magnetic field suggest the possibility that these systems will fit with classical thermal radiation. It may be possible to give a full scattering calculation in the case of cyclotron motion in order to provide the needed test of the connection between conformal symmetry and classical thermal radiation.

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1. Introduction

The blackbody radiation problem remains unsolved within classical physics. Although the introduction of energy quanta a century ago has led to the currently accepted explanation within quantum theory, there is still no firm conclusion as to whether or not blackbody radiation can be explained within classical physics. In this paper we discuss the current situation and introduce new arguments which again [1] suggest that the classical solution requires the restriction to purely electromagnetic systems where conformal symmetry is involved.

A century ago, the mismatch between mechanics and electromagnetism was clearly evident. Traditional classical mechanics is invariant under Galilean symmetry transformation whereas Maxwell's equations are invariant under Lorentz transformation. Also, traditional classical mechanics contains no scales or fundamental constants, whereas classical electromagnetism contains several fundamental constants, including a limiting speed of light

in vacuum c , a smallest electronic charge e , and Stefan's blackbody radiation constant a_S . It follows that classical mechanics allows separate scalings of length, time and energy, whereas classical electromagnetism allows only a single scaling which couples together the scales of length, time and energy in conformal symmetry. Although the mismatch between mechanics and electromagnetism in connection with relativity has been decided in favour of the Lorentz transformation of electromagnetism being the more fundamental, there has been no such consensus regarding scaling. In the twentieth century, the mismatch between mechanics and electromagnetism led to the development of a new mechanics, quantum mechanics, which ties together the scaling of time and energy through Planck's constant \hbar while still allowing any mechanical potential to enter the theory. Here we again suggest the alternative resolution to the mismatch which regards classical electromagnetism as the more fundamental theory. Perhaps the restriction to 'mechanical' systems which appear from electromagnetic sources holds the key to a classical understanding of blackbody radiation and also of at least some parts of atomic theory. This suggestion has recently been bolstered by Cole and Zou's simulation work obtaining the hydrogen ground state within classical physics [2]¹.

In order to describe as much of nature as possible, classical electromagnetic theory must include classical electromagnetic zero-point radiation as the homogeneous boundary condition for Maxwell's equations [3]. Zero-point radiation is random radiation which is homogeneous in space, isotropic in direction, and invariant under Lorentz transformation. Furthermore, it turns out that the zero-point radiation spectrum is the unique spectrum invariant under conformal transformation [4]. The invariance requirements determine the spectrum up to a multiplicative constant as an energy \mathcal{U} per normal mode given by $\mathcal{U} = \text{const} \times \omega$ where ω is the angular frequency of the radiation mode. In order to reproduce the experimentally observed van der Waals forces, the constant must be chosen as approximately $\text{const} = 0.525 \times 10^{-34} \text{ J s}$, recognizable as $\text{const} = (1/2)\hbar$ where \hbar has the magnitude of Planck's constant.

Now if random classical zero-point radiation is present in the classical electromagnetic theory at zero temperature, then all the ideas of traditional classical statistical mechanics are invalid, except as high-temperature limits. (Just such a situation is also found in quantum theory.) Also, if zero-point radiation is present, it is possible to give derivations of the Planck spectrum of thermal radiation using classical physics from several different points of view: energy equipartition for translational degrees of freedom [5], thermal fluctuations above zero-point radiation [6], comparisons between diamagnetic and paramagnetic behaviour [7], the acceleration of point electromagnetic systems through zero-point radiation [8], and maximum entropy ideas connected with Casimir forces [9].

However, despite these derivations, the problem of classical radiation equilibrium is not completely solved by the introduction of classical electromagnetic zero-point radiation because the Planck spectrum with zero-point radiation can be shown to be unstable under scattering by charged nonlinear mechanical systems. In 1924 van Vleck [10] solved the Fokker–Planck equation for a general class of charged nonlinear mechanical systems in random radiation (in the electric dipole approximation and small-charge limit) with the conclusion that the mechanical system achieved equilibrium with random radiation only when the mechanical system was distributed according to the Boltzmann distribution and the radiation corresponded to the Rayleigh–Jeans spectrum. There were further calculations [11, 12] coming to this same conclusion in the 1970s and 1980s, including heroic calculations by Blanco *et al* [13] regarding a relativistic charged particle in certain classes of mechanical potentials.

¹ Cole and Zou's numerical simulation calculations suggest that classical zero-point radiation acting on the classical model for hydrogen (a point charge in a Coulomb potential) produces a steady-state probability distribution which is finite and which seems to approach the ground state obtained from solution of the Schrödinger equation for a point charge in the Coulomb potential. Cole and Zou have no adjustable parameters in their calculations.

These scattering calculations would seem to settle the matter in the negative from the most fundamental point of view. After all, radiation equilibrium means the stability of the radiation spectrum under scattering by a charged particle. However, there is just one failure of the scattering calculations—all of them excluded scattering by a Coulomb potential or indeed by any purely electromagnetic system.

Here we are suggesting that this exclusion contains the very essence for understanding classical radiation equilibrium. All of the scattering calculations to date involve potentials which contradict the conformal-related scaling symmetries of thermal radiation. We suggest that only when purely electromagnetic scattering systems are used can we hope to obtain the thermal radiation spectrum observed in nature.

2. Outline of the paper

The outline of our discussion is as follows. We begin by noting the restrictive form of scaling symmetry which follows from conformal symmetry and which holds in electromagnetism. We then confirm the validity of this scaling for thermal radiation. Next we discuss the contrasting nonrelativistic scattering calculations which appear in the literature. It is pointed out that the nonrelativistic nonlinear scattering systems do not retain a thermodynamic distribution under an adiabatic compression. Thus they do not allow a derivation of the Wien displacement law. We suggest that such systems merely illustrate the mismatch between classical mechanics and electromagnetism. When we turn to two purely electromagnetic systems in thermal radiation at $T = 0$, in zero-point radiation, we find that both systems have very special relativistic properties which suggest possibilities for thermal equilibrium which are lacking in other mechanical scattering systems. Finally we give a closing summary.

3. Discussion of fundamental constants

Nonrelativistic classical mechanics has no fundamental constants. Thus there is no preferred length, time or energy, nor any fundamental connection among them. Accordingly we may choose independent scales of length, of time and of energy, and indeed the commonly used systems of units reflect this independence. Thus given any nonrelativistic mechanical system, a second system may be constructed which has twice the spatial dimensions, three times the speed, and four times as much energy. In contrast, nature associates three fundamental constants with the observed solutions of Maxwell's electromagnetic theory. These are the largest speed of electromagnetic waves c , the smallest amount of charge e , and Stefan's constant a_S of blackbody radiation. Each of these constants is left unchanged under transformations of the conformal group, the largest group of transformations which leaves invariant Maxwell's equations.

The three constants c , e and a_S connect together the scales of length, time and energy which enter electromagnetism. Thus the fundamental speed c joins the scales of length and time. The elementary charge e couples the scales of energy and distance through the potential energy U of two elementary charges separated by a distance r , $U = e^2/r$. Stefan's constant a_S couples energy density u to temperature T ; however, since the temperature scale is not fundamental, we may consider Stefan's constant a_S divided by Boltzmann's constant k_B to the fourth power so that Stefan's law connecting total thermal energy density u to the energy $k_B T$ of long-wavelength modes can be rewritten as $u = (a_S/k_B^4)(k_B T)^4$. Evidently Stefan's constant (in the form a_S/k_B^4) again connects the energy scale and the length scale. Indeed, since Stefan's constant can be reexpressed [14] in terms of Planck's constant \hbar , $a_S/k_B^4 = \pi^2/(15\hbar^3 c^3)$, we could just as well have chosen c , e and \hbar as our fundamental constants of electromagnetic

theory rather than c , e and a_S . We have chosen to start with Stefan's constant a_S , which was introduced in 1879, rather than Planck's constant $h = 2\pi\hbar$, which was introduced in 1900, so as to emphasize that the fundamental constants arise in connection with solutions of classical electromagnetic theory and need not have any connection with ideas of energy quanta.

Maxwell's equations are conformal invariant [15], retaining exactly the same form under a conformal transformation; the solutions of Maxwell's equations are conformal covariant in the sense that under conformal transformation one solution of Maxwell's equations is mapped into another solution of Maxwell's equations. Rather than working with the full conformal group, it is convenient here to consider only one small part of this group, the dilatations. (Invariance under dilatations and Lorentz transformation implies conformal invariance.) Dilatations are the mappings where all lengths, times and energies are multiplied by a constant σ , $\mathbf{r} \rightarrow \mathbf{r}' = \sigma\mathbf{r}$, $t \rightarrow t' = \sigma t$ and $U \rightarrow U' = (1/\sigma)U$. This arrangement preserves the fundamental electromagnetic constants— c with units of length divided by time $r/t = \sigma r/(\sigma t) = r'/t'$, e with units of the square root of energy times length $(Ur)^{1/2} = (U/\sigma)^{1/2}(\sigma r)^{1/2} = (U'r')^{1/2}$, and \hbar with units of energy times time $Ut = (U/\sigma)\sigma t = U't'$. Since the scaling couples together length, time and energy, we will term this ' $\sigma_{lE^{-1}}$ scale-invariance'.

We note that nature's solutions to the homogeneous Maxwell's equations couple the scales of length, time and energy but contain no special length, time or energy. Thus an electromagnetic plane wave in vacuum of frequency ν , and wavelength λ , and electric field amplitude E_0 can also be reinterpreted by an observer using a dilated scale of length, time and energy as a plane wave of frequency $\nu' = (1/\sigma)\nu$, wavelength $\lambda' = \sigma\lambda$ and amplitude $E'_0 = (1/\sigma^2)E_0$. Similarly, blackbody radiation at temperature $k_B T$ in a box of volume V , with total thermal energy $U = (a_S/k_B^4)V(k_B T)^4$ and entropy $S = (4/3)(a_S/k_B^4)V(k_B T)^3$ would be reinterpreted by an observer using a dilated scale of length, time and energy as blackbody radiation at temperature $k_B T' = (1/\sigma)k_B T$, in a volume $V' = (\sigma^3)V$, with total energy $U' = a_S k_B^4 V' (k_B T')^4 = a_S k_B^4 (\sigma^3 V) (k_B T/\sigma)^4 = U/\sigma$ and total entropy $S' = (4/3)a_S k_B^4 V' (k_B T')^3 = (4/3)a_S k_B^4 (\sigma^3 V) (k_B T/\sigma)^3 = S$. Thus the entropy of thermal radiation is unchanged by any $\sigma_{lE^{-1}}$ scaling transformation.

The $\sigma_{lE^{-1}}$ scale-invariance of electromagnetism can be continued from Maxwell's equations over to classical electron theory with point masses, provided mass m is scaled as $m \rightarrow m' = m/\sigma$, corresponding to the scaling for energy $U = mc^2$. Indeed Haantjes has shown [16] that the conformal invariance of electromagnetism can be extended to classical electron theory provided we transform a point mass as $m \rightarrow m' = m/\sigma(x)$ where $\sigma(x)$ is the space-time-dependent scale factor of the conformal transformation.

Maxwell's equations and, indeed, thermal radiation satisfy $\sigma_{lE^{-1}}$ symmetry where the scale factor $\sigma_{lE^{-1}}$ is an arbitrary real number, $0 < \sigma_{lE^{-1}} < \infty$. If we consider a change of units corresponding to $\sigma_{lE^{-1}}$ scaling, then any parameter (such as volume) which changes under $\sigma_{lE^{-1}}$ must also have a continuous range of values 0 to ∞ , and is appropriate for adiabatic change in mechanics and thermodynamics, except in the case of mass where we usually think in terms of substitution rather than of continuous change of a parameter. Indeed, the scaling $m \rightarrow m' = m/\sigma$ implies that all masses are available in the theory $0 < m < \infty$, and that there is no special value for mass. This aspect is not observed in nature (e.g., the mass of the electron is indeed special) and is regarded here as an aspect beyond our present electromagnetic considerations.

4. Linear and nonlinear oscillator scatterers

Around the year 1900, Planck [17] considered a very small charged harmonic oscillator in interaction with random electromagnetic radiation. The random classical radiation is

expressed as a superposition of plane waves with random phases, extending throughout space. The radiation provides a stationary random process for the electric field at any spatial point, and any scattering system will be driven into random oscillation with an amplitude of motion which is again a stationary random process. Planck showed that in the small-charge limit, a small linear oscillator (electric dipole approximation for the radiation interaction) had the same average energy as a radiation mode of the same frequency. Since the harmonic oscillator treated in the electric dipole approximation does not scatter the radiation into any second frequency, it comes to equilibrium with *any* isotropic spectrum of random radiation. Clearly, the small harmonic oscillator does not determine the equilibrium spectrum of thermal radiation based upon its scattering. Now if classical Boltzmann statistical mechanics is applied to the oscillator, then one finds energy equipartition for the oscillator and the Rayleigh–Jeans spectrum for the radiation with which the oscillator is in equilibrium. However, we have noted that Boltzmann statistical mechanics cannot be valid in any classical system which includes the classical electromagnetic zero-point radiation which is present in nature. Thus the derivation of the Rayleigh–Jeans spectrum from classical Boltzmann statistical mechanics is irrelevant to a description of nature, except as a high-temperature limit.

Of far more interest are derivations of radiation equilibrium from nonlinear scattering calculations. A nonlinear oscillator which exchanges energy with several frequencies will indeed enforce a radiation equilibrium; in general, it will absorb net radiation energy at one frequency and emit net radiation energy at a different frequency. The energy of the oscillator is balanced, but for a general radiation spectrum the random radiation is not in equilibrium since there is a continual transfer of energy from one frequency to another. A scattering calculation for a small nonlinear oscillator [11] shows that this scatterer pushes a general radiation spectrum towards the Rayleigh–Jeans spectrum. This fits with van Vleck’s work [10] of 1924 using a Fokker–Planck equation for the behaviour of a general class of nonlinear oscillators. van Vleck showed that nonrelativistic nonlinear oscillators treated in the dipole approximation and small-charge limit come to equilibrium at the Boltzmann distribution for the oscillators and the Rayleigh–Jeans spectrum for radiation. These calculations are made using nonrelativistic physics and so include the possibility of particle velocities exceeding the speed of light c ; however, since the electric dipole approximation is used for the interaction with radiation, only the frequencies of the motion and not the velocities are relevant for the interaction with radiation, and hence no contradiction with relativity becomes apparent. Indeed, one can consider relativistic mechanics for the particle motion in a general class of potentials, and, as Blanco *et al* showed [13], one still arrives at basically the same uncomfortable conclusion involving a balance for Boltzmann statistics and the Rayleigh–Jeans spectrum.

What the class of potentials considered in the nonlinear scattering calculations do not satisfy is conformal symmetry. Conformal symmetry suggests a tight connection between frequency, energy, and spatial extent, and this tight connection is relevant to the interaction with radiation. The use of relativistic particle mechanics is indeed sufficient to guarantee that the particle speed does not exceed the speed of light, but this does not come close to the restrictions of conformal symmetry. For example, the nonlinear oscillator considered in the scattering calculations [11, 12] of 1976 and 1978 involves the same mechanical system treated by Born [18]. Solving the Hamiltonian

$$H = p^2/(2m) + m\omega_0^2 x^2/2 + \Gamma x^3/3 \quad (1)$$

with a perturbative solution in the nonlinearity governed by the parameter Γ , we find

$$x = D_1 \sin w + D_2(3 + \cos 2w), \quad w = \omega t + \phi, \quad (2)$$

where the amplitudes D_1 and D_2 can be written in terms of the action variable J as

$$D_1 = [2J/(m\omega_0)]^{1/2} \quad \text{and} \quad D_2 = -\Gamma\omega_0 J / (3\omega_0^4 m^2). \quad (3)$$

The Hamiltonian and oscillation frequency can also be rewritten as

$$W = J\omega_0 - 5\Gamma^2(\omega_0 J)^2 / (12\omega_0 m^3) \quad \text{and} \quad \omega = \omega_0 - 5\Gamma^2 J / (6\omega_0^4 m^3) \quad (4)$$

The strength of the nonlinearity Γ determines the ratio of the amplitudes D_1 and D_2 of the first and second harmonics which determines whether a little or a lot of radiation is exchanged between ω and 2ω going into the electric dipole radiation mode labelled by $l = 1, m = 0$. Thus the ratio of the radiation energy absorbed and emitted at the fundamental and its second harmonic is freely adjustable through the arbitrary nonlinear coupling constant Γ . Such arbitrariness does not exist for electromagnetic systems satisfying conformal symmetry.

5. Wien displacement law and the mismatch with the Boltzmann distribution

In addition to the arbitrariness in the radiation connection for nonlinear oscillators, there is a second troubling aspect in connection with the Wien displacement law. Thermodynamics within the context of classical theory leads to the Stefan–Boltzmann law $u = a_S T^4$ and to Wien's displacement law $\mathcal{U}(\omega, T) = \omega f(\omega/T)$, where u is the thermal energy per unit volume, a_S is Stefan's constant, $\mathcal{U}(\omega, T)$ is the energy per normal mode of (angular) frequency ω and temperature T , and $f(\omega/T)$ is an unknown function. Both these laws are experimentally observed to hold in nature. The derivation of Wien's displacement law depends upon carrying out a quasi-static change in the system [19], usually an adiabatic compression of the radiation which maintains the radiation as a thermal spectrum at a smoothly changing temperature. The reflection of the radiation from a slowly-moving reflecting surface on a piston is one method of carrying out the adiabatic compression [20].

Now radiation inside a reflecting-walled cavity cannot bring itself to the thermal equilibrium spectrum. Rather, there must be some scattering system which changes the spectrum of radiation. As noted above, using small nonlinear oscillator scattering systems, we find that equilibrium occurs for the Boltzmann distribution for the mechanical system and the Rayleigh–Jeans law for the radiation spectrum. Following in the spirit of the traditional derivation of the Wien displacement theorem, it is interesting to consider an adiabatic change where both the radiation and the scattering systems are changed. We can consider an ensemble of many identical nonlinear mechanical scattering systems in thermal equilibrium with random radiation. The radiation acts as a heat bath for the ensemble of nonlinear oscillators. If we now regard the mechanical scattering systems as decoupled from the radiation, we can carry out adiabatic changes separately for the mechanical oscillators and for the radiation. Now adiabatic compression for thermal radiation (when the shape of the container is unchanged) is equivalent to a $\sigma_{lT} E^{-1}$ scale change. Thus any spectrum of radiation with an energy $\mathcal{U}(\omega, T) = \omega f(\omega/T)$ for a mode of frequency ω is carried into

$$\mathcal{U}'(\omega', T') = \omega' f(\omega'/T') = \frac{\omega}{\sigma} f\left(\frac{\omega/\sigma}{T/\sigma}\right) = \frac{1}{\sigma} \omega f\left(\frac{\omega}{T}\right) = \frac{1}{\sigma} \mathcal{U}(\omega, T) \quad (5)$$

and specifically, the Rayleigh–Jeans spectrum at temperature T is carried into the Rayleigh–Jeans spectrum at $T' = T/\sigma$. On the other hand, mechanics tells us that under a change of parameter of a mechanical system, the action variables J do not change [21]. Thus the probability distribution $P(J, T, b)$ for the action variables of the mechanical ensemble at temperature T and parameter b does not change when the parameter b is slowly changed to b' . *Linear* scattering systems with no harmonics $H = J\omega$ (which were used in the suggestive classical derivations [5–9] of the Planck spectrum) are indeed transformed into distributions

which are again in equilibrium with the thermal radiation spectrum at some new temperature T' . Indeed Cole [22] has used this behaviour of linear oscillator systems to give a derivation of the Wien displacement theorem. Thus for linear oscillators $H = J\omega$,

$$P(J, T, \omega) = \text{const} \exp \left[-\frac{H}{\omega f(\omega/T)} \right] = \text{const} \exp \left[-\frac{J}{f(\omega/T)} \right] \quad (6)$$

becomes

$$P(J, T, \omega) = \text{const} \exp \left[-\frac{J}{f(\omega/T)} \right] = \text{const} \exp \left[-\frac{J}{f(\omega'/T')} \right] = P(J, T', \omega'). \quad (7)$$

The linear oscillator keeps a thermal distribution but at a new frequency and new temperature, $\omega/T = \omega'/T'$. However, for nonlinear oscillators, an adiabatic change in some mechanical parameter takes the ensemble of mechanical systems away from the Boltzmann distribution. Thus for Born's nonlinear oscillator mentioned above, a change in the parameter Γ does not preserve a Boltzmann distribution. There is no choice of temperature T' for which

$$P(J, T, \Gamma) = \text{const} \exp \left[-\frac{H}{k_B T} \right] = \text{const} \exp \left[-\frac{J\omega_0 - \{5\Gamma^2(\omega_0 J)^2/(12\omega_0 m^3)\}}{k_B T} \right] \quad (8)$$

equals

$$P(J, T', \Gamma') = \text{const} \exp \left[-\frac{\{J\omega_0 - 5\Gamma'^2(\omega_0 J)^2/(12\omega_0 m^3)\}}{k_B T'} \right] \quad (9)$$

for all J if $\Gamma \neq \Gamma'$. After the adiabatic mechanical transformation, the nonlinear oscillators are no longer in equilibrium with the Rayleigh–Jeans spectrum at any new temperature. It seems surprising indeed that the scattering system which is supposed to bring radiation to equilibrium cannot maintain the equilibrium under any adiabatic change. This suggests that these mechanical systems may not be allowed systems in classical radiation physics. It is symptomatic of the mismatch between mechanics and electromagnetism [23]².

6. Adiabatic changes and zero-point radiation

We have suggested that mechanical systems which do not satisfy conformal symmetry are not suitable for discussing classical radiation equilibrium. We have seen that they involve excessive freedom in their connections with radiation and also do not behave appropriately under adiabatic changes of parameters. At this point we need to show that there are indeed mechanical scattering systems for radiation which overcome these objections. In this section, we will limit our attention to temperature $T = 0$ where only zero-point radiation is present. We have already remarked that zero-point radiation is the unique spectrum of random radiation which is invariant under conformal transformation [4]. We suggest that purely electromagnetic scattering systems (which are related to conformal symmetry) will not scatter zero-point radiation towards a new spectrum but will give radiation equilibrium at temperature $T = 0$.

We emphasize that allowed systems should not exchange energy with zero-point radiation during adiabatic changes. Now the zero-point radiation is invariant under any adiabatic change. However, when a mechanical parameter is changed adiabatically, the mechanical system takes on a new average energy and a new frequency pattern. The mechanical system and radiation

² Even within traditional classical statistical mechanics, it seems surprising that an adiabatic change of a mechanical parameter does not in general maintain the Boltzmann distribution for mechanical systems. To my knowledge, the only place where the matter is even hinted at is in a paper by Szilard. Szilard notes that some special systems (such as blackbody radiation) retain the 'normal' (that is thermodynamic) distribution under adiabatic compression, but that general mechanical systems do not. General systems must be rethermalized repeatedly during adiabatic change, see [23].

(in the small-charge limit) can be regarded as two separate thermodynamic systems which can be brought into contact through the electric charge. An average exchange of energy between the mechanical system and the radiation during an adiabatic change suggests a change in entropy, and at $T = 0$ the ideas of thermodynamics suggest that no changes of entropy are possible.

Some aspects of this problem were explored [24] in 1978. It was pointed out that all the small nonrelativistic mechanical systems without harmonics behaved appropriately under changes of mechanical parameters in zero-point radiation. In zero-point radiation the distribution of action variables for these systems takes the form

$$P(J) = \frac{1}{\hbar/2} \exp \left[-\frac{J}{\hbar/2} \right] \quad (10)$$

and has no dependence upon any mechanical parameters. Such systems include point harmonic oscillator systems in several dimensions and in magnetic fields, and also nonrelativistic cyclotron motion for a charge in a magnetic field. The scattering systems described in the earlier work are treated in the electric dipole approximation and interact with radiation at single frequencies without coupling to any harmonics. Thus there is no exchange of radiation between radiation modes of different frequency, and hence no radiation equilibrium is forced by the mechanical scattering systems.

However, this is a limiting approximation made for small systems when the speed of the particle is close to zero. Any finite-velocity motion by a charged particle entails the emission of radiation at all the harmonics of the fundamental frequency with a distribution of radiated energy among the harmonics which is determined by the parameter $\beta = v/c$. For example, a charged particle e moving in the xy -plane in a circle of radius r with speed $v = c\beta$ gives a power-radiated per unit solid angle at angle θ from the z -axis at the n th harmonic at frequency $n\bar{\omega} = nv/r$ in the form [25]

$$\frac{dP_n}{d\Omega} = \frac{e^2 \bar{\omega}^4 r^2}{2\pi c^3} n^2 \left\{ \left[\frac{dJ_n(n\beta \sin \theta)}{d(n\beta \sin \theta)} \right]^2 + \frac{\cot^2 \theta}{\beta^2} J_n^2(n\beta \sin \theta) \right\}. \quad (11)$$

The particle can also absorb energy at each harmonic if there is energy present in the radiation field. Thus all classical electromagnetic systems of finite size interact with many frequencies and hence determine a spectrum of radiation equilibrium.

It must be emphasized just how different is this finite-size mechanism for equilibrium from that involved in point nonlinear mechanical oscillators. For charged nonlinear scatterers treated in the dipole approximation, the equilibrium is forced by the mechanical system with its connection between harmonics depending upon some arbitrary nonlinear parameter. For Born's nonlinear oscillator mentioned earlier, Γ is the nonlinear parameter. The nonlinear mechanical oscillator contains within itself the ratios of the amplitudes for the harmonics with no reference to the relative speed $\beta = v/c$ of the particle and the radiation. On the other hand, the finite size of purely harmonic motions gives an electromagnetic basis for forcing equilibrium. For uniform circular motion, the relative speed $\beta = v/c$ of the particle and the radiation completely determines the relative power emitted into the various harmonics.

In addition to forcing an equilibrium radiation spectrum, finite-size systems have new possibilities for their distributions $P(J, \omega_0)$ of action variables in zero-point radiation. We must determine whether the adiabatic invariance of the distribution $P(J)$ which held in zero-point radiation in the nonrelativistic limit continues for the full relativistic treatment.

7. Aspects of scattering by relativistic cyclotron motion

7.1. Equations of motion

The simplest purely electromagnetic scattering system of which we are aware is cyclotron motion, the circular motion of a charged particle in a uniform magnetic field. Here we wish to point out some of the aspects of scattering by this conformally covariant system. We hope to complete and report on a full scattering calculation in the not distant future.

When we ignore the connection to radiation, cyclotron motion of a particle of charge e and mass m in a uniform magnetic field \mathbf{B} (in the lab frame) is described by the Hamiltonian

$$H = \sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 c^2 + m^2 c^4} \quad \text{where} \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{B} \times \mathbf{r}}{2c}. \quad (12)$$

The equations of motion follow as

$$m \frac{d}{dt} \left(\frac{\mathbf{v}}{(1 - v^2/c^2)^{1/2}} \right) = e \frac{\mathbf{v}}{c} \times \mathbf{B} \quad (13)$$

or

$$\frac{d^2 \mathbf{r}}{d\tau^2} = \frac{d\mathbf{r}}{d\tau} \times \vec{\omega}_0 \quad \text{with} \quad \vec{\omega}_0 = e\mathbf{B}/(mc) \quad (14)$$

where τ is the particle proper time

$$d\tau = dt/\gamma = \sqrt{1 - v^2/c^2} dt. \quad (15)$$

We note from equation (14) that (independent of the orbital radius and velocity) the orbital rotation rate is always ω_0 when measured using the particle's proper time. Taking the charge e as positive and the magnetic field in the negative z -direction, the solutions correspond to uniform circular motion at frequency $\bar{\omega} = \omega_0/\gamma$ with $\gamma = (1 - v^2/c^2)^{-1/2}$,

$$x - x_0 = r \cos[\bar{\omega}t + \phi] \quad y - y_0 = r \sin[\bar{\omega}t + \phi] \quad \text{where} \quad \bar{\omega} = \omega_0/\gamma. \quad (16)$$

The angular momentum J , including both mechanical and electromagnetic field angular momentum, is an adiabatic invariant³

$$J = m\gamma vr - \frac{eBr^2}{2c} = \frac{1}{2}m\gamma vr = \frac{eBr^2}{2c} \quad (17)$$

where the last two forms follow from the equation of motion for the circular orbit, $m\gamma v^2/r = evB/c$. The action variable J determines the orbit radius and also the orbit velocity as

$$r = \sqrt{\frac{2J}{m\omega_0}} \quad \beta = \sqrt{\frac{2J\omega_0/(mc^2)}{1 + 2J\omega_0/(mc^2)}} \quad \gamma = \sqrt{1 + \frac{2J\omega_0}{mc^2}} \quad (18)$$

and these expressions hold for all J and ω_0 . As J ranges over the interval $(0, \infty)$, the velocity parameter β (or γ) is a monotonically increasing function of J for every choice of m or ω_0 . There is no preferred value of β or γ .

³ See [25, pp 588–9]. We are using $1/(2\pi)$ times Jackson's action variable.

7.2. Nonrelativistic limit of cyclotron motion

In the limit of nonrelativistic motion in the lab frame, $2J\omega_0/(mc^2) \ll 1$, cyclotron motion in classical zero-point radiation was treated [24, 26] in 1978 and 1980. In this limit, all the motion is at the single frequency $\omega_0 = eB/(mc)$, since $\bar{\omega} \rightarrow \omega_0$ as $\gamma \rightarrow 1$. A Fokker–Planck equation was obtained for the probability distribution $P(J)$ of the action variable J in a random radiation spectrum with energy per normal mode $\mathcal{U}(\omega) = (1/2)\hbar\omega$ corresponding to zero-point radiation. The Fokker–Planck equation gave the probability distribution of equation (10). This distribution seems exactly appropriate for zero-point radiation; even when the magnetic field B is changed, and hence the nonrelativistic frequency $\omega_0 = eB/(mc)$ is changed, the mechanical motion remains in equilibrium with the zero-point radiation and does not exchange any energy (on average) with the zero-point radiation. All changes of average mechanical energy when the magnetic field B is changed are due to the Faraday-induced electric field associated with the changing B .

7.3. Relativistic treatment of cyclotron motion

However, what happens when we go to the full relativistic treatment? In the relativistic treatment, the mechanical motion varies in frequency with J since $\bar{\omega} = \omega_0/\gamma$, and also the charge interacts with radiation at all the harmonics of the mechanical frequency. Do we still have radiation equilibrium? Do we still have $P(J)$ independent of ω_0 in order to maintain our ideas of entropy under adiabatic changes of magnetic field at temperature $T = 0$?

Now relativistic cyclotron motion is an electromagnetic system satisfying conformal invariance. Thus we expect that the conformal-invariant zero-point radiation spectrum is maintained and that the distribution $P(J)$ maintains the form given in equation (10). In order to prove this we need a complete calculation of the radiation scattering. However, for the present we will present some suggestive evidence. The Fokker–Planck equation needed to obtain $P(J)$ requires calculations of the radiation energy loss per unit time by the mechanical particle motion, the average energy absorbed per unit time from zero-point radiation, and the average of the square of the energy absorbed per unit time from zero-point radiation. It is easy to calculate the radiated energy per unit time for a charged particle e moving in a circle of radius r with frequency $\bar{\omega}$ as

$$P_{\text{emitted}} = \frac{2}{3} \frac{e^2}{c^3} \bar{\omega}^4 \gamma^4 r^2 \quad (19)$$

where $\gamma = (1 - v^2/c^2)^{-1/2} = [1 - (r\bar{\omega}/c)^2]^{-1/2}$. In the nonrelativistic limit, the radiation emission for cyclotron motion is

$$P_{\text{emitted}}^{\text{cyclotron NR}} = \frac{2}{3} \frac{e^2}{c^3} \omega_0^4 r^2 \quad (20)$$

since in the nonrelativistic limit $\gamma \rightarrow 1$ and $\bar{\omega} = \omega_0/\gamma \rightarrow \omega_0$. But now note when we substitute the fully relativistic expressions $\bar{\omega} = \omega_0/\gamma$ and $r = \sqrt{2J/(m\omega_0)}$ for cyclotron motion into the fully relativistic expression for P_{emitted} , we find

$$P_{\text{emitted}}^{\text{cyclotron R}} = \frac{2}{3} \frac{e^2}{c^3} \bar{\omega}^4 \gamma^4 r^2 = \frac{2}{3} \frac{e^2}{c^3} \omega_0^4 \frac{2J}{m\omega_0} \quad \text{for all } J \text{ and } \omega_0. \quad (21)$$

This expression is identical with the nonrelativistic expression. When written in terms of J and ω_0 , the expression makes no explicit reference to any velocity and retains its nonrelativistic form for *all* values of J and ω_0 . If we go to the inertial frame in which the charge is at rest at some instant, then for a small time interval the particle motion is nonrelativistic and the charge is found moving in circular arcs with the same frequency $\omega_0 = eB/(mc)$ as is involved

in nonrelativistic motion and with the same J as is involved in the lab motion. Also, the zero-point radiation spectrum is Lorentz invariant and hence the same in any inertial frame. Thus in the momentarily comoving reference frame, where the motion is nonrelativistic, cyclotron motion seems to take the same form as for nonrelativistic motion in the lab frame. This suggests the possibility that relativistic cyclotron motion will maintain the same distribution $P(J)$ in equation (10) which is invariant under adiabatic changes, exactly as required for our ideas of thermodynamic equilibrium at $T = 0$.

Furthermore, the connection between the orbit and the relative energy radiated into various harmonics is not at our disposal, as it is in the nonlinear oscillator case, but rather is tightly connected to formulae involving spherical Bessel functions. This suggests the possibility that this purely electromagnetic system will allow equilibrium with zero-point radiation.

7.4. Relativistic limit of the harmonic potential

In order to emphasize that cyclotron motion has very special properties not encountered with nonelectromagnetic systems, we can consider relativistic motion in a harmonic oscillator potential in the lab frame, when limiting ourselves to circular orbits. In the nonrelativistic limit, motion in a harmonic oscillator potential $V_{\text{SHO}}(r) = (1/2)kr^2$ is at the frequency $\omega_0 = \sqrt{k/m}$ and involves no harmonics. In terms of relativistic particle mechanics for a circular orbit in this same potential,

$$m\gamma \frac{v^2}{r} = kr \quad J = m\gamma vr. \quad (22)$$

Combining these expressions gives

$$\gamma^3 \beta^4 = \left(\frac{J\omega_0}{mc^2}\right)^2 \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \beta = \frac{\bar{\omega}r}{c} \quad \bar{\omega} = \frac{\omega_0}{\gamma^{1/2}}. \quad (23)$$

Solving the equation connecting β and J , we find that

$$\beta \approx \sqrt{\frac{J\omega_0}{mc^2}} \quad r \approx \left(\frac{J}{m\omega_0}\right)^{1/2} \quad \text{for} \quad \frac{J\omega_0}{mc^2} \ll 1 \quad (24)$$

and

$$\gamma \approx \left(\frac{J\omega_0}{mc^2}\right)^{2/3} \quad r \approx \left(\frac{Jc}{m\omega_0^2}\right)^{1/3} \quad \text{for} \quad \frac{J\omega_0}{mc^2} \gg 1. \quad (25)$$

Then in terms of the parameter $J\omega_0/(mc^2)$, radiation emission is given by

$$P_{\text{emission}}^{\text{circular SHO}} \approx \frac{2}{3} \frac{e^2}{c^3} (c\omega_0)^2 \left(\frac{J\omega_0}{mc^2}\right) \quad \text{for} \quad \frac{J\omega_0}{mc^2} \ll 1 \quad (26)$$

while

$$P_{\text{emission}}^{\text{circular SHO}} \approx \frac{2}{3} \frac{e^2}{c^3} (c\omega_0)^2 \left(\frac{J\omega_0}{mc^2}\right)^2 \quad \text{for} \quad \frac{J\omega_0}{mc^2} \gg 1. \quad (27)$$

The change from linear over to quadratic dependence on J shows clearly that $P_{\text{emission}}^{\text{circular SHO}}$ does not retain its nonrelativistic functional form for the harmonic oscillator potential.

Indeed we note the relation $\bar{\omega} = \omega_0/\gamma^{1/2}$ which holds for the harmonic oscillator potential is not connected to particle proper time. If we go to the momentarily comoving reference frame in this case, the rotation frequency in this frame depends upon the speed of the particle in its orbit in the lab frame; it does *not* take the nonrelativistic harmonic oscillator value $\omega_0 = \sqrt{k/m}$. Thus in zero-point radiation, the energy pick-up and loss in the instantaneous

rest frame of the particle depends upon a frequency which varies with the velocity of the particle in the lab frame. The pick-up and loss of energy in the momentarily comoving reference frame of the particle do not take the same form as for nonrelativistic motion. Relativistic mechanical motions in nonelectromagnetic systems do not have characteristics suitable for thermodynamic equilibrium with zero-point radiation.

8. Comments on the Coulomb potential

In the previous section, we focused our attention on cyclotron motion because this seems the simplest electromagnetic scattering system; cyclotron motion has an easily-calculable nonrelativistic limit for large mass m , and in zero-point radiation seems to retain its nonrelativistic forms at all velocities when expressed in terms of J and ω_0 . However, despite its complications, the Coulomb potential allows some interesting observations [27].

The Coulomb potential $V_C(r) = e^2/r$ is the only potential of the form $V(r) = k/r^n$ where the constant k giving the strength of the potential does not change under a $\sigma_{tE^{-1}}$ scale transformation. Since an energy must transform as $1/\sigma$, $V' = (1/\sigma)V$, we have

$$V'(r') = k'/r'^n = k'/(\sigma^n r^n) = (k'/\sigma^n)/r^n = (1/\sigma)k/r^n = (1/\sigma)V(r) \quad (28)$$

so that only for $n = 1$ do we have $k' = k$. Thus the electronic charge e appearing in the Coulomb potential $V_C(r) = e^2/r$ is invariant under conformal transformation and no other potential-strength constant is so invariant.

Since for non-Coulomb potentials the constant k changes with the choice of scale $\sigma_{tE^{-1}}$, k must be treated as a parameter subject to variation $0 < k < \infty$, and can be used to carry out adiabatic changes in the mechanical system. Using such adiabatic changes, it may well be possible to transfer energy from one frequency range to another in the presence of zero-point radiation, hence violating our ideas of entropy changes at temperature $T = 0$. On the other hand for the Coulomb potential, the strength parameter e^2 is scale invariant and hence is not subject to adiabatic change.

The Hamiltonian for a mass m in the Coulomb potential $H = (p^2 c^2 + m^2 c^4)^{1/2} + e^2/r$ can be rewritten in terms of action-angle variables as [28]⁴

$$H = mc^2 \left(1 + \frac{(e^2/c)^2}{[(J'_3 - J'_2) + \sqrt{J_2'^2 - (e^2/c)^2}]^2} \right)^{-1/2}. \quad (29)$$

We note that $H/(mc^2)$ involves only the action variables J'_2, J'_3 and the quantity e^2/c . In the presence of zero-point radiation (which is scale invariant and indeed invariant under conformal transformation), the only scale for length, time or energy is through the mass m , and the pattern of velocities for given J values is the same independent of the mass m . Thus for a point charge in a Coulomb potential in zero-point radiation, we cannot obtain a nonrelativistic limit by considering a large-mass limit. Rather, if one can find the solution for this classical hydrogen atom for one choice of the mass m , the same distribution $P(J)$ will hold for any other mass, while all lengths, times and energies will be rescaled, and the velocities will be left unchanged.

Cyclotron motion and Coulomb potential motion in zero-point radiation are very different. Cyclotron motion depends upon the mass m and the pure number $J\omega_0/(mc^2) = JeB/(m^2 c^3)$ where $0 < J < \infty$. In the presence of zero-point radiation, all average quantities depend upon the mass m and $\hbar\omega_0/(mc^2)$ in the small-charge limit. Here there is no reason for a preferred choice of the value of \hbar . However, for the Coulomb potential, the Hamiltonian form (29) in

⁴ See [21, p 498]. We are using action variables which are $1/(2\pi)$ times those in Goldstein's text.

terms of action variables shows that e^2/c is a lower bound [28] for the action variable J'_2 so that we require $e^2/c < J'_2 < \infty$. Now the values of the action variables in zero-point radiation are dependent upon the multiplicative constant \hbar giving the scale of the zero-point radiation. Thus this suggests the basis for a connection between e^2/c and \hbar . If \hbar is too small, then the value of J'_2 will be too close to the cut-off e^2/c which appears in the relativistic mechanics of the Coulomb potential. We again suggest [1] that a full understanding of the behaviour of a charged particle in the Coulomb potential in classical zero-point radiation will lead to a calculation of the fine structure constant.

9. Closing summary

In this work we revisit the suggestion that scattering by classical electromagnetic systems (which involve conformal symmetry) will provide an explanation for the Planck spectrum for thermal radiation within the context of classical physics. This time we go beyond the considerations of scaling symmetry which were mentioned 15 years ago. We suggest that the several calculations of radiation scattering using nonlinear mechanical systems merely illustrate the mismatch between mechanics and electromagnetism and are not relevant for understanding nature. We point out the curious fact that most mechanical systems do not preserve the Boltzmann distribution under adiabatic change of a parameter. This fact seems at variance with our expectations in connection with derivations of Wien's displacement theorem where we expect a scatterer which enforces an equilibrium spectrum to remain in equilibrium during a suitable adiabatic change. Linear oscillators do not enforce radiation equilibrium in the nonrelativistic approximation, but indeed do impose equilibrium when treated relativistically. We emphasize some of the striking properties of charged particle motion in a Coulomb potential or in a uniform magnetic field which suggest the possibility that these systems will fit with classical thermal radiation. In particular, cyclotron motion involves linear motion in the nonrelativistic approximation and has surprising continuities in form when treated relativistically, and the Coulomb potential is unique in not allowing adiabatic changes of the potential-strength parameter e . Finally we note that it may be possible to give a full scattering calculation in the case of cyclotron motion which should provide a crucial test of the suggested connection between conformal symmetry and classical thermal radiation.

Awareness of the mismatch between mechanics and electromagnetism seems to involve contrasting perspectives between relativistic invariance and conformal invariance. In the last decades of the nineteenth century, physicists became concerned about the mismatch between mechanics and electromagnetism in connection with the fundamental constant c , the unique value of the speed of light in vacuum appearing in nature. In the early years of the twentieth century, the relativistic symmetry of Maxwell's equations and its solutions was recognized, and the constant c was taken as a fundamental connection between the scales of length and time. Within classical physics, there has been no comparable attention to the mismatch between mechanics and electromagnetism reflected in the fundamental electronic charge e and Stefan's constant a_S (or equivalently Planck's constant \hbar) which occur in the solutions to Maxwell's equations which appear in nature. At present these constants which couple energy and length are not usually associated with the conformal invariance of Maxwell's equations discovered by Cunningham and Bateman in 1909.

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